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$1/d$-expansions for the free energy of weakly embedded site animal models of branched polymers

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Abstract. We give the $1/d$-expansion, to order $1/d^5$, for the limiting reduced free energy of a weakly embedded two-variable site animal model of branched polymers. Hence, we are able to derive expansions for the free energy of related one-variable models and the growth constant of weakly and strongly embedded trees, again to order $1/d^5$. We argue that the free energy expansions are asymptotically correct for a small range of fugacities only. Exact results show that animals with a compact hypercubic structure make the overwhelming contribution to the total number of weakly embedded site animals, especially for large d. This result is not reflected in the free energy $1/d$-expansions, where trees make the dominant contribution.

We also derive new exact enumeration data for lattice animals using intermediate calculations in the derivation of the $1/d$-expansions. Thus, we give partition functions for the general d-dimensional hypercubic lattice for one to 13 sites.

1. Introduction

A $1/d$-expansion is an asymptotic expansion of some function $f(d)$ in powers of $1/d$, where d is the dimension. They were first introduced by Fisher and Gaunt (1964) for the growth constant $\mu(d)$ of self-avoiding walks (SAWs), amongst other things. The existence of this expansion was proved rigorously by Kesten (1964) who showed that it is asymptotic rather than convergent (see also Gerber and Fisher 1974, Fisher and Singh 1990, Hara and Slade 1995). Other $1/d$-expansions relating to SAWs have since been obtained. The expansion for the growth constant of neighbour-avoiding walks (NAWs) was given by Gaunt *et al* (1984) through order $1/d^3$ and an expansion for the critical amplitude, A, of SAWs was given through order $1/d^5$ by Gaunt (1986).

More recently, Nemirovsky *et al* (1992a, b) and Ishinabe *et al* (1994) used $1/d$-expansions to study an *interacting* SAW model of linear polymers. They used a nearest-neighbour contact energy $-\epsilon$ and found the limiting reduced free energy, F, the amplitude, A, and the end-to-end distance through order $1/d^5$. The coefficients of the $1/d$-expansions are now temperature-dependent functions of $z = e^{\epsilon/kT}$. The results for SAWs (Fisher and Gaunt 1964, Gaunt 1986) may be recovered by setting $z = 1$, while setting $z = 0$ extends the expansion of Gaunt *et al* (1984) for NAWs through $1/d^5$.

A $1/d$-expansion for lattice animals was first given by Gaunt *et al* (1976) for the growth constant, Λ_s, of strongly embedded site animals through order $1/d^2$. Gaunt and Ruskin (1978) presented the equivalent expansion for the growth constant, λ_b, of weakly embedded bond animals through order $1/d^2$, and this was corrected and extended by Harris (1982) through order $1/d^5$. The $1/d$-expansions for the growth constants of weakly and strongly embedded lattice trees, λ_0 and Λ_0, respectively, were found by Gaunt *et al* (1982) through order $1/d^2$.

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There have been two more recent studies of $1/d$-expansions for lattice animals. Gaunt *et al* (1994) gave expansions for the growth constants of weakly embedded site animals, λ_s, and strongly embedded bond animals, Λ_b, both through order $1/d^3$. The expansions for λ_0, Λ_0 and Λ_s were also extended through order $1/d^3$. Rigorous methods also established the ordering

$$\lambda_s > \lambda_b > \lambda_0 > \Lambda_s \geqslant \Lambda_b > \Lambda_0 \tag{1.1}$$

though the strict inequality $\Lambda_s > \Lambda_b$ is a conjecture. The $1/d$-expansions for the growth constants support this ordering, as do numerical estimates even for the lowest dimensions.

Peard and Gaunt (1995) derived and studied $1/d$-expansions for the limiting reduced free energies of two models of *interacting* lattice animals, the k- and k'-models. These models introduce a nearest-neighbour contact fugacity e^β into the weakly embedded site and bond animal ensembles, respectively, and are the lattice animal analogues of the model studied by Nemirovsky *et al* (1992a, b) and Ishinabe *et al* (1994). The $1/d$-expansions for the free energies, $F^{(d)}(\beta; k)$ and $F^{(d)}(\beta; k')$, are now functions of d and β, and were calculated through order $1/d^5$. The $1/d$-expansions for $\ln\lambda_s$, $\ln\lambda_b$, $\ln\Lambda_s$ and $\ln\Lambda_b$ were easily deduced from $F^{(d)}(0; k)$, $F^{(d)}(0; k')$, $F^{(d)}(-\infty; k)$ and $F^{(d)}(-\infty; k')$, respectively, again through order $1/d^5$. The expansion for $\ln\lambda_b$ agreed with Harris (1982) through all orders and the other expansions agree with Gaunt *et al* (1994) through order $1/d^3$. Peard and Gaunt (1995) also found that the free energy $1/d$-expansions were not asymptotically valid for any $\beta > \beta_0 \geqslant 0$, where β_0 is unknown but is unlikely to be greater than the collapse temperature $\beta_c(d)$.

In this paper, we will study the $1/d$-expansion for the limiting reduced free energy of the $\{c, k\}$-model, a weakly embedded two-variable interacting site animal model. Two-variable models provide both useful and interesting ways of studying lattice animals (see, for example, Peard and Gaunt 1995, Janse van Rensburg 2000). The $1/d$-expansion for a two-variable model of an interacting uniform star polymer has been derived and studied by Gaunt and Yu (2000). In section 2, we will define the $\{c, k\}$-model and show how the free energy of this model is related to six independent one-variable models which have been discussed elsewhere in the literature (Madras *et al* 1990, Flesia and Gaunt 1992, Flesia *et al* 1994, Gaunt and Flesia 1990, 1991). In section 3, we will calculate the $\{c, k\}$-model free energy expansion through order $1/d^5$ and hence deduce the $1/d$-expansions for the growth constants λ_0 and Λ_0 through the same order. $1/d$-expansions for the other one-variable model free energies can also be found using the relations in section 2. The intermediate calculations in the derivation of the $1/d$-expansions also allow us to derive new exact enumeration data, which we will discuss at the end of section 3. In section 4, we will use exact results to show that the expansions are asymptotically correct only for a small range of fugacities and we will give an explanation why this is so. In particular, we argue that the $1/d$-expansion for λ_s is incorrect. Finally, in section 5, we summarize our results.

2. Models

A *lattice animal* is a connected graph on a lattice consisting of occupied lattice vertices, *sites*, connected by occupied lattice edges, *bonds*. We can measure the size of an animal by the number of sites n or bonds b and we will refer to this as the *site* or *bond content*, respectively. Further, when animal size is measured by site content, we will call the animals *site animals*, with an analogous definition for *bond animals*. (It should be noted that other authors have defined site and bond animals differently, especially in the context of percolation clusters.) The site and bond content of an animal allows us to define two different ways of taking the thermodynamic limit; by letting $n \to \infty$ or $b \to \infty$. In this paper, we will be concerned with site animals only and hence the $n \to \infty$ limit. Bond animals will be considered elsewhere.

There are other useful parameters defining the conformation of a lattice animal. The *cyclomatic index*, c, is the maximum number of bonds that can be removed without disconnecting the animal. It is connected to n and b by Euler's relation

$$c = b - n + 1. \tag{2.1}$$

The number of nearest-neighbour pairs of sites with no connecting bond of the animal is the number of *contacts* k, while p is the total number of nearest-neighbour *pairs* of sites. A *solvent contact* is defined as an edge of the lattice which joins a site of the animal to a neighbouring unoccupied lattice vertex. Let the number of solvent contacts be s. Finally, we use q to denote the usual edge *percolation perimeter*. The above quantities are related by

$$p = b + k \tag{2.2}$$

$$s = \lambda n - 2b - 2k \tag{2.3}$$

$$q = s + k \tag{2.4}$$

where λ is the coordination number of the lattice. The set of animals with c fixed are called *c-animals* and if $c = 0$ the c-animals are commonly called *trees*. Euler's relation (2.1) shows site and bond content are equivalent ways of measuring tree or c-animal size. If $k = 0$, the animal (or tree) is a section graph of the lattice and is said to be *strongly embedded*. Animals with no restriction on k are subgraphs of the lattice and are said to be *weakly embedded*.

In the literature, many lattice animal models of branched polymers have been defined using the above parameters. Following Flesia (1993), we may consider all these models together by defining a generic ω-model with the partition function

$$Z_n(\beta; \omega) = \sum_{\omega} a_n(\omega)\, \mathrm{e}^{\beta\omega} \tag{2.5}$$

where $a_n(\omega)$ is the number of weakly embedded lattice animals with n sites and $\omega \in \{c, k, b, p, q, s\}$ cycles, contacts, etc. This gives us six one-variable models for weakly embedded site animal models, which we will refer to (using lowercase letters) as the c-model, k-model, etc, depending on the parameter used in the Boltzmann weight. The generic limiting reduced free energy is defined by

$$F(\beta; \omega) = \lim_{n\to\infty} \frac{1}{n} \ln Z_n(\beta; \omega). \tag{2.6}$$

It is easy to show that not all the models are independent. In particular, we find

$$F(\beta; c) = F(\beta; b) - \beta \tag{2.7}$$

$$F(\beta; s) = F(-2\beta; p) + \lambda\beta. \tag{2.8}$$

This leaves four independent weakly embeddable site animal models, which we choose to be the c-, k-, s- and q-models.

We can define similar models for the other objects that we have considered; strongly embedded site animals, weakly and strongly embedded trees and c-animals. However, by definition, all strongly embedded objects have $k = 0$ and all trees and c-animals have c fixed. This means that, with n fixed, relations (2.1)–(2.4) leave just one independent parameter, say c for strongly embedded objects and k for trees and c-animals. In turn, this means that the number of independent models we can define for these objects is limited. We will choose the

C-model (uppercase indicating strong embeddings) and the t_c-models as independent. They are defined by the partition functions

$$Z_n(\beta; C) = \sum_c A_n(c)\, e^{\beta c} \tag{2.9}$$

$$Z_n(\beta; t_c) = \sum_k t_{n,c}(k)\, e^{\beta k} \tag{2.10}$$

respectively, where $A_n(c)$ is the number of strongly embedded site animals with n sites and c cycles, and $t_{n,c}(k)$ is the number of weakly embedded c-animals with n sites and k contacts. We can reduce the number of independent models still more. Flesia *et al* (1992a) proved that the limiting reduced free energies of the t_c-models were all equal, i.e. $F(\beta; t_c) = F(\beta; t_0) \equiv F(\beta; t)\ \forall c$. Thus, for simplicity, we will choose the contact model for trees, the t-model, as the independent model.

All these one-variable models, the c-, k-, s-, q-, C- and t-models, can be related to a single two-variable model. Given fixed n, relations (2.1)–(2.4) allow only two independent parameters, say c and k. This suggests that we define a two-variable $\{c, k\}$-model partition function thus,

$$Z_n(\beta_1, \beta_2; c, k) = \sum_{c,k\geqslant 0} a_n(c, k)\, e^{\beta_1 c+\beta_2 k}. \tag{2.11}$$

$a_n(c, k)$ is the number of weakly embedded site animals with n sites, c cycles and k contacts, and e^{β_1} and e^{β_2} are cycle and contact fugacities, respectively. The limiting reduced free energy of the $\{c, k\}$-model is

$$F(\beta_1, \beta_2; c, k) = \lim_{n\to\infty} \frac{1}{n} \ln Z_n(\beta_1, \beta_2; c, k) \tag{2.12}$$

and is related to the six one-variable model free energies as follows:

$$F(\beta; c) = F(\beta_1 = \beta, \beta_2 = 0; c, k) \tag{2.13}$$

$$F(\beta; k) = F(\beta_1 = 0, \beta_2 = \beta; c, k) \tag{2.14}$$

$$F(\beta; s) = F(\beta_1 = -2\beta, \beta_2 = -2\beta; c, k) + (\lambda - 2)\beta \tag{2.15}$$

$$F(\beta; q) = F(\beta_1 = -2\beta, \beta_2 = -\beta; c, k) + (\lambda - 2)\beta \tag{2.16}$$

$$F(\beta; t) = \lim_{\beta_1\to-\infty} F(\beta_1, \beta_2 = \beta; c, k) \tag{2.17}$$

$$F(\beta; C) = \lim_{\beta_2\to-\infty} F(\beta_1 = \beta, \beta_2; c, k). \tag{2.18}$$

In the derivation of (2.17) and (2.18), we assumed that the limits $\beta_1, \beta_2 \to -\infty$ and $n \to \infty$ commute. Thus, the free energy $F(\beta_1, \beta_2; c, k)$ of the two-variable $\{c, k\}$-model allows us to study the thermodynamics of all six interacting one-variable site animal models.

We note that other authors have preferred to work with different two-variable models. Thus, Flesia *et al* (1992b, 1994) defined the $\{s, k\}$-model and the $\{s, c\}$-model partition functions. Using (2.1) and (2.3), we find

$$s + 2k + 2c = (\lambda - 2)n + 2. \tag{2.19}$$

Given this relationship between c, k and s (for fixed n), we can relate the free energy of the $\{s, k\}$-model, as well as the free energy of the $\{s, c\}$-model, to the free energy (2.12) of the $\{c, k\}$-model.

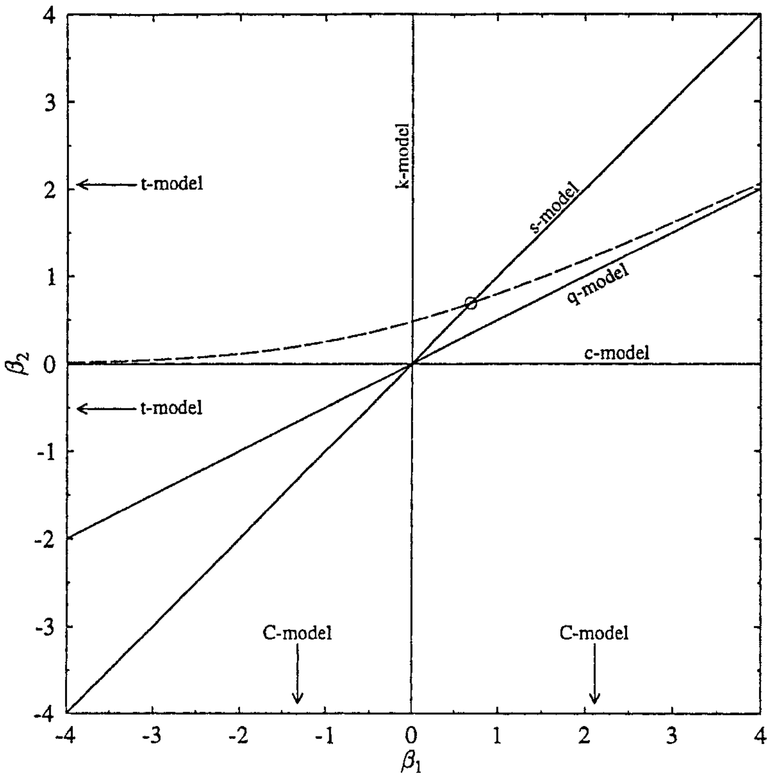

Figure 1. (β_1, β_2)-plane of the $\{c, k\}$-model and relations to the one-variable site animal models. The percolation line is shown by the broken curve and the percolation point is indicated by $\odot$.

The relations we have derived between the one- and two-variable models are illustrated in figure 1, which shows the (β_1, β_2)-plane of the $\{c, k\}$-model. The one-variable models are shown as lines and limits on this plane. The figure is valid in all dimensions and for all lattices.

It is possible to derive a connection between the bond percolation problem and lattice animal models of branched polymers. Flesia *et al* (1992b, 1994) used this connection to deduce the transition temperature of the s-model on the square lattice in the context of the $\{s, k\}$-model. The same result is readily derived starting from the $\{c, k\}$-model. Thus, one finds that the *percolation curve* along which the $\{c, k\}$-model is isomorphic to the bond percolation problem is defined parametrically by the equations

$$e^{\beta_1} = p/(1-p)^2 \qquad e^{\beta_2} = 1/(1-p).$$

For the square lattice, $p_c = \frac{1}{2}$ (see Essam 1972), and therefore $\beta_1^c = \beta_2^c = \ln 2$. Since this point, indicated by the symbol $\odot$ in figure 1, is on the s-line it corresponds to a singularity in the s-model. Equation (2.15) then gives the critical temperature of the s-model on the square lattice as $\beta_c^s = -\frac{1}{2}\ln 2 = -0.346\ldots$.

3. $1/d$-expansions

A full account of the derivation of the $1/d$-expansion for $F(\beta; k)$ was given by Peard and Gaunt (1995). In this section, we outline the derivation of the $1/d$-expansion for the free energy of the two-variable $\{c, k\}$-model.

First, we rewrite (2.11) so that the dependence on the lattice dimension is explicit. Thus, for a d-dimensional simple hypercubic lattice of coordination number $\lambda = 2d = \sigma + 1$, we have

$$Z_n(y, z; c, k) = \sum_{i=1}^{n-1} \sum_{c,k \geqslant 0} f_{i,c,k}^{(n)} y^c z^k \binom{d}{n-i} \qquad n \geqslant 2 \tag{3.1}$$

where $y = e^{\beta_1}$ and $z = e^{\beta_2}$ are cycle and contact fugacities, respectively. The coefficients $f_{i,c,k}^{(n)}$ are the number of animals with n sites, c cycles and k contacts, spanning an $(n-i)$-dimensional subspace of the lattice. For $n \leqslant 8$, they are calculated from exact enumeration data alone and are given in appendix A. (Exact enumeration data are given in the Appendices of Madras *et al* (1990) for the square and simple cubic lattices, and have been extended through $n \leqslant 21$ and $n \leqslant 19$, respectively. Our only other exact enumeration data not derivable from appendix A are for $n = 14$ and 15 when $d = 4$. We will present all our additional exact enumeration data in a future publication.)

Next, we find the functions of n, $f_{i,c,k}(n)$, which generate the sequences $f_{i,c,k}^{(n)}$ for fixed i, c and k. For example, from appendix A we see that the sequence $\{f_{1,0,0}^{(n)}\}$ is $\{1, 4, 32, 400, 6912, 153\,664, 4\,194\,304, \ldots; n \geqslant 2\}$ and is generated by

$$f_{1,0,0}(n) = 2^{n-1} n^{n-3}. \tag{3.2}$$

This function counts trees with n sites that span $(n-1)$ dimensions (Peard and Gaunt 1995) and since all these clusters have zero cycles and zero contacts, it is clear that

$$f_{1,c,k}(n) = 0 \qquad \forall\, k, c > 0. \tag{3.3}$$

It is convenient to define new functions of n, $g_{i,c,k}(n)$, by the relation

$$f_{i,c,k}(n) = 2^{n-2i+1} n^{n-2i-1} g_{i,c,k}(n). \tag{3.4}$$

For $i = 1$, it follows from (3.2) to (3.4) that the only non-zero g-function is $g_{1,0,0} = 1$. For larger values of i, the non-zero $g_{i,c,k}(n)$ are all assumed to be *polynomials* in n. They may be derived from exact enumeration data and the constraints implied by the existence of a $1/d$-expansion for the free energy. (For further details, but in the context of the k-model, see Peard and Gaunt (1995).) The non-zero $g_{i,c,k}(n)$ for $i \leqslant 6$ are given in appendix B. (The corresponding polynomials for the $\{c', k'\}$-model, or bond animal version of our model, are given in appendix 3 of Peard and Gaunt (1995) for $i \leqslant 5$.) A useful check on the polynomials $g_{i,c,k}(n)$ is provided by noticing that

$$\sum_{c \geqslant 0} g_{i,c,k}(n) = g_{i,k}(n) \tag{3.5}$$

where $g_{i,k}(n)$ are the corresponding polynomials for the k-model and are listed for all $i \leqslant 6$ in appendix 2 of Peard and Gaunt (1995). Furthermore,

$$\sum_{c,k \geqslant 0} g_{i,c,k}(n) = g_i(n) \tag{3.6}$$

where $2^{n-2i+1} n^{n-2i-1} g_i(n)$ is the coefficient of $\binom{d}{n-i}$ in the expansion corresponding to (3.1) for weakly embedded site animals (see, for example, Gaunt *et al* 1994, equation (3.2)).

From (3.1) and (3.4), we obtain our final expression for the partition function of the $\{c, k\}$-model, namely

$$Z(n, y, z; c, k) = \sum_{i=1}^{n-1} \sum_{c,k \geqslant 0} 2^{n-2i+1} n^{n-2i-1} g_{i,c,k}(n)\, y^c z^k \binom{d}{n-i} \qquad n \geqslant 2. \tag{3.7}$$

This expression is valid for all $n \geqslant 2$ and all d, an unusual result which allows us to derive new exact enumeration data (see later in this section).

To calculate a $1/d$-expansion for the free energy, we first expand the binomial coefficients in (3.7) in inverse powers of σ using equation (2.17) of Peard and Gaunt (1995). Then formally taking the logarithm of $Z(n, y, z; c, k)$, dividing by n and letting $n \to \infty$, we obtain the central result of this paper, namely, the asymptotic development

$$F^{(d)}(y, z; c, k) = \lim_{n\to\infty} \frac{1}{n} \ln Z(n, y, z; c, k) \tag{3.8}$$

$$\begin{aligned}
&= \ln\sigma + 1 + \left(-\tfrac{5}{2} + \tfrac{1}{2}y + 2z\right)\sigma^{-1} + \left(-\tfrac{13}{6} - \tfrac{3}{2}y + \tfrac{3}{8}y^2 - 5z + \tfrac{3}{2}yz + \tfrac{9}{2}z^2\right)\sigma^{-2} \\
&\quad + \left(-\tfrac{191}{12} + \tfrac{27}{4}y - \tfrac{41}{8}y^2 + \tfrac{13}{12}y^3 + 33z - 28yz + \tfrac{89}{12}y^2z \right.\\
&\quad \left. - \tfrac{135}{2}z^2 + \tfrac{91}{4}yz^2 + \tfrac{130}{3}z^3\right)\sigma^{-3} \\
&\quad + \left(-\tfrac{139}{5} - \tfrac{115}{2}y + 45y^2 - \tfrac{39}{2}y^3 + \tfrac{457}{192}y^4 + \tfrac{1}{6}y^5 - \tfrac{499}{2}z + \tfrac{535}{2}yz - \tfrac{525}{4}y^2z \right.\\
&\quad + \tfrac{52}{3}y^3z + 2y^4z + 679z^2 - \tfrac{785}{2}yz^2 + \tfrac{213}{4}y^2z^2 + 11y^3z^2 - \tfrac{1501}{2}z^3 + 104yz^3 \\
&\quad \left. + \tfrac{106}{3}y^2z^3 + \tfrac{953}{4}z^4 + 68yz^4 + 64z^5\right)\sigma^{-4} \\
&\quad + \left(-\tfrac{1585}{3} + \tfrac{2687}{16}y - \tfrac{7919}{24}y^2 + \tfrac{6097}{24}y^3 - \tfrac{2747}{32}y^4 + \tfrac{3661}{480}y^5 + \tfrac{3}{4}y^6 + \tfrac{3597}{4}z \right.\\
&\quad - 1798yz + \tfrac{43\,301}{24}y^2z - 742y^3z + \tfrac{6049}{80}y^4z + \tfrac{19}{2}y^5z - 4522z^2 + \tfrac{44\,353}{8}yz^2 \\
&\quad - \tfrac{11\,339}{4}y^2z^2 + \tfrac{2713}{8}y^3z^2 + \tfrac{111}{2}y^4z^2 + \tfrac{62\,831}{6}z^3 - 6397yz^3 \\
&\quad + \tfrac{21\,775}{24}y^2z^3 + 197y^3z^3 - \tfrac{20\,333}{2}z^4 + \tfrac{26\,823}{16}yz^4 + \tfrac{939}{2}y^2z^4 + \tfrac{56\,321}{20}z^5 \\
&\quad \left. + 772yz^5 + 752z^6\right)\sigma^{-5} + \mathrm{O}\left(\sigma^{-6}\right).
\end{aligned} \tag{3.9}$$

Relations (2.13)–(2.18) immediately give the $1/d$-expansions for the free energy of the one-variable models. For convenience, these are given in appendix C, except for $F^{(d)}(z; k)$, obtained by setting $y = 1$, which agrees with expansion (2.20) given in Peard and Gaunt (1995) through all orders. The expansions for the growth constants $\ln\lambda_s$, $\ln\Lambda_s$, $\ln\lambda_0$ and $\ln\Lambda_0$ are found from (3.9) by letting $y = z = 1$, $y = 1$ and $z = 0$, $y = 0$ and $z = 1$, and $y = z = 0$, respectively. The expansions for $\ln\lambda_s$ and $\ln\Lambda_s$ are identical to those given in Peard and Gaunt (1995). The expansions for $\ln\lambda_0$ and $\ln\Lambda_0$ are

$$\ln\lambda_0^{(d)} = \ln\sigma + 1 - \tfrac{1}{2}\sigma^{-1} - \tfrac{8}{3}\sigma^{-2} - \tfrac{85}{12}\sigma^{-3} - \tfrac{931}{20}\sigma^{-4} - \tfrac{2777}{10}\sigma^{-5} + \mathrm{O}\left(\sigma^{-6}\right) \tag{3.10}$$

and

$$\ln\Lambda_0^{(d)} = \ln\sigma + 1 - \tfrac{5}{2}\sigma^{-1} - \tfrac{13}{6}\sigma^{-2} - \tfrac{191}{12}\sigma^{-3} - \tfrac{139}{5}\sigma^{-4} - \tfrac{1585}{3}\sigma^{-5} + \mathrm{O}\left(\sigma^{-6}\right) \tag{3.11}$$

which extend the expansions given through order σ^{-3} by Gaunt *et al* (1994).

As indicated earlier, partition function (3.7) allows us to calculate new exact enumeration data. The initially available exact enumeration data, all animals through $n = 21, 19, 15, 13, 13, 11$ for $d = 2$ through $d = 7$, respectively, enable us to calculate a certain number of the coefficients $f_{i,c,k}^{(n)}$. In fact, we know all non-zero $f_{i,c,k}^{(n)}$ for $n \leqslant 8$ but fewer and fewer $f_{i,c,k}^{(n)}$ as n increases. In turn, the known coefficients allow us to find all non-zero generating functions $f_{i,c,k}(n)$ for $i \leqslant 3$ and fewer and fewer of these as i increases. However, when we calculate the $1/d$-expansion, we impose the constraint that $F(y, z; c, k)$ is independent of n and this allows us to calculate previously unknown generating functions $f_{i,c,k}(n)$. Then, from partition function (3.7), we can calculate previously unknown coefficients

$f_{i,c,k}^{(n)}$. In fact, we can find all non-zero $f_{i,c,k}^{(n)}$ through $n = 13$ and hence obtain all the partition functions (3.1) through $n = 13$. These are given in appendix A.

This result is unusual. The expressions equivalent to $f_{i,c,k}(n)$ for other problems, such as self-avoiding walks (Fisher and Gaunt 1964, Nemirovsky *et al* 1992a) and uniform star polymers (Gaunt *et al* 1984, Gaunt and Yu 2000), are valid only for $n \geqslant n_0$, where $n_0 \equiv n_0(i)$, not $n \geqslant 2$. This means that we cannot calculate new exact enumeration data for these problems by the above method. The underlying reason for this is unknown but it is notable that the functionality for self-avoiding walks and star polymers is fixed, while the functionality of lattice animals varies between 1 and λ.

4. Discussion

It is generally believed that $1/d$-expansions are asymptotically correct (Fisher and Gaunt 1964, Kesten 1964) in the sense that if $F^{(d)}(\beta)$ is the asymptotic development of $F(\beta)$ in powers of $1/d$, then

$$\lim_{d\to\infty} \left[F^{(d)}(\beta) - F(\beta)\right] = 0. \tag{4.1}$$

Thus, as $d \to \infty$, $F^{(d)}(y, z; c, k)$ should give increasingly accurate estimates of the free energy $F(y, z; c, k)$. However, this is not true for all temperatures of the k-model. Peard and Gaunt (1995) have studied the free energy, $F(z; k)$, of this model by comparing estimates derived from the $1/d$-expansion, $F^{(d)}(z; k)$, with known rigorous bounds (Madras *et al* 1990) and series estimates (Flesia and Gaunt 1992). The $1/d$-estimates agree with the series estimates and lie within the rigorous bounds for $d \geqslant 6$ and $z < 1$. For $z \gg 1$, the series and $1/d$-estimates are not consistent with the rigorous bounds for any d, although the estimates still agree with each other. Clearly, this behaviour is inconsistent with an asymptotic expansion in powers of $1/d$. The situation is further complicated by the fact that the rigorous bounds depend on the growth constants $\ln\lambda_s = F(z = 1; k)$ and $\ln\Lambda_s = F(z = 0; k)$, which are themselves estimated either from series or $1/d$-expansions.

In this section, we will argue that the true value of $\ln\lambda_s$ lies well away from the series and $1/d$-estimates given in Peard and Gaunt (1995): certainly for $d \geqslant 6$, almost certainly for $d = 5$ and 4, and perhaps for $d = 3$. Only when $d = 2$ is the estimate of $\ln\lambda_s$ likely to be accurate. On the other hand, we argue that the series and $1/d$-estimates of $\ln\lambda_0$, $\ln\Lambda_s$ and $\ln\Lambda_0$ are likely to be accurate for all d. Hence, we will argue that the $1/d$-expansion, $F^{(d)}(y, z; c, k)$, is asymptotically correct only in the expanded phase.

Madras *et al* (1990) gave rigorous bounds for the limiting reduced free energy of the c-model, equations (3.40) and (3.44), and hence, by setting $\beta = 0$, rigorous bounds on $\ln\lambda_s$, namely,

$$(d-1)\ln 2 \leqslant \ln\lambda_s \leqslant \ln\lambda_0 + (d-1)\ln 2. \tag{4.2}$$

Let λ_B be the growth constant for trees on the Bethe lattice, namely

$$\begin{aligned}\ln\lambda_B &= \sigma\ln\sigma - (\sigma-1)\ln(\sigma-1)\\ &= \ln\sigma + 1 - \sum_{k=1}^{\infty}\frac{1}{k(k+1)}\sigma^{-k}.\end{aligned} \tag{4.3}$$

Equation (3.53) of Madras *et al* (1990) implies that

$$\ln\lambda_b \leqslant \frac{(d-1)}{d}\beta - \ln(1 - \mathrm{e}^{-\beta/2d}) \tag{4.4}$$

for any $\beta > 0$. This inequality is most effective when $\beta = 2d\ln[(2d-1)/(2d-2)] > 0$. Substituting this value of β into (4.4) and using the sequence of inequalities (1.1) shows that

$$\ln\lambda_0 < \ln\lambda_b \leqslant \ln\lambda_B. \tag{4.5}$$

Substituting (4.5) into (4.2) gives the rigorous bounds

$$(d-1)\ln 2 \leqslant \ln\lambda_s < \ln\lambda_B + (d-1)\ln 2. \tag{4.6}$$

We can extend this result by dividing by d and letting $d \to \infty$ to give

$$\lim_{d\to\infty} \frac{1}{d}\ln\lambda_s = \ln 2 \tag{4.7}$$

or

$$\ln\lambda_s \sim d\ln 2 \qquad d \to \infty. \tag{4.8}$$

The exact result (4.8) disagrees with the $1/d$-expansion (Peard and Gaunt 1995), namely

$$\ln\lambda_s \sim \ln\sigma + 1 = \ln(2d-1) + 1 \qquad d \to \infty. \tag{4.9}$$

Further, Peard and Gaunt (1995) showed the agreement between series and $1/d$-estimates of $\ln\lambda_s$ improves with dimension. For instance, in $d = 3$, series and $1/d$-estimates give $\ln\lambda_s^{(s)} = 2.434\pm0.004$ and $\ln\lambda_s^{(\sigma)} = 2.500\pm0.018$, respectively, while in $d = 6$ the respective estimates are $\ln\lambda_s^{(s)} = 3.371 \pm 0.013$ and $\ln\lambda_s^{(\sigma)} = 3.373 \pm 0.001$. In $d = 6$, our rigorous lower bound (4.2) gives $\ln\lambda_s > 3.46$, so the series and $1/d$-estimates of Peard and Gaunt (1995), although in good agreement, are definitely wrong when $d = 6$. The same conclusion is true for all $d \geqslant 6$; the series and $1/d$-expansions give the result $\ln\lambda_s \sim \ln(2d-1)+1$, while we know rigorously that $\ln\lambda_s \sim d\ln 2$.

This error occurs because the exact enumeration data and $1/d$-expansions do not fully represent the contribution of lattice animals with a compact hypercubic structure to the lattice animal ensemble. To understand this, consider a hypercube of $n_h = m^d$ sites. We will find upper and lower bounds for the number of animals, a_{m^d}, spanning all m^d sites. First, we note the binomial expansion

$$n_h = m^d = \sum_{i=0}^{d} 2^i (m-2)^{d-i}\binom{d}{i} \tag{4.10}$$

which partitions the hypercube into sets of points, the ith term giving the number of sites in the $(d-i)$th set. From this, we can calculate the number of bonds, $b_h = d(m^d - m^{d-1})$, in the hypercube and the number of cycles, $c_h = 1 + (d-1)m^d - dm^{d-1}$. We now construct a tree spanning all m^d sites of the hypercube. This can be done by deleting c_h bonds in the hypercube without disconnecting the animal. The number of bonds in the tree is thus $b_t = m^d - 1$ and the number of contacts is equal to the number of cycles in the original hypercube,

$$k_t = 1 + (d-1)m^d - dm^{d-1}. \tag{4.11}$$

We now add $1, 2, \ldots, k_t$ bonds to each of the k_t contacts to create $\binom{k_t}{1}, \binom{k_t}{2}, \ldots, \binom{k_t}{k_t}$ new, distinguishable animals from the hypercube. Similarly, we can start with m^d occupied sites with no connecting bonds and add $1, 2, \ldots, b_h$ bonds to the unoccupied contacts. This will create a set of objects $\binom{b_h}{1}, \binom{b_h}{2}, \ldots, \binom{b_h}{b_h}$ which includes all the animals that can be created from a hypercube of m^d sites. Thus,

$$2^{k_t} \equiv \sum_{i=0}^{k_t}\binom{k_t}{i} \leqslant a_{m^d} \leqslant \sum_{i=0}^{b_h}\binom{b_h}{i} \equiv 2^{b_h}. \tag{4.12}$$

Taking logarithms, dividing by $n = m^d$ and letting $m \to \infty$ gives

$$(d-1)\ln 2 \leqslant \lim_{m\to\infty} \frac{1}{m^d} \ln a_{m^d} \leqslant d \ln 2. \tag{4.13}$$

If we divide by d and let $d \to \infty$, we find

$$\lim_{d\to\infty} \frac{1}{d} \lim_{m\to\infty} \frac{1}{m^d} \ln a_{m^d} = \ln 2. \tag{4.14}$$

Comparing results (4.7) and (4.14), we see that animals with a compact hypercubic structure make the overwhelming contribution to $\ln \lambda_s$ in the $d \to \infty$ limit.

Animals with a compact hypercubic structure appear in the exact enumeration only if the data include animals with $n > 2^d$ sites. This is true for $d = 2$ and 3 with the exact enumeration data available. A stricter requirement of $n > 3^d$ sites would exclude the simple cubic exact enumeration data as well. So, the series estimates of Peard and Gaunt (1995) are underestimating $\ln \lambda_s$ because the exact enumeration data lack enough terms; certainly for $d \geqslant 6$, almost certainly for $d = 5$ and 4 and perhaps for $d = 3$.

However, none of this explains why the $1/d$-expansion for $\ln \lambda_s$ is failing; it was after all derived using the correct formal procedure. The problem is that, in deriving the $1/d$-expansion, the limit $d \to \infty$ should be taken after the limit $n \to \infty$. In the formal procedure normally used, the order of the limits is interchanged. In this case, the dominating structures in the $1/d$-expansion are necessarily those which span the full d-dimensions. If tree-like structures dominate the set of configurations, this interchange of limits should be justifiable. Indeed, in the case of self-avoiding walks, it is known (Fisher and Gaunt 1964, Hara and Slade 1995) that the order of the limits is irrelevant. On the other hand, in the case of site animals, where we have argued that compact structures make the overwhelming contribution, taking the limits in the correct order is crucial.

Now that we have understood why the $1/d$-expansion for $\ln \lambda_s$ is failing, we will try to find the range of validity of our free energy expansions. Clearly, if $F^{(d)}(y = 1, z = 1; c, k) \equiv \ln \lambda_s^{(d)}$ is failing, $F^{(d)}(y, z; c, k)$ will fail when both $y > 1$ and $z > 1$. In this region, animals with many cycles and/or contacts will contribute ever greater weights to $Z_n^{(d)}(y, z; c, k)$, and we know these animals are not properly represented.

Having shown that the expansion for $\ln \lambda_s$ is not asymptotically correct, we will argue that the expansions for the other growth constants are likely to be asymptotically correct. According to (4.5), $\ln \lambda_B \geqslant \ln \lambda_b$. This implies that trees, not compact hypercubic animals, make the overwhelming contribution to $\ln \lambda_b$ and, from the inequalities (1.1), that trees make the overwhelming contribution to λ_0, Λ_s and Λ_0 as well. A comparison of the convergent expansion for $\ln \lambda_B$, equation (4.4), and the $1/d$-expansions for $\ln \lambda_0$ (equation (3.10)), $\ln \Lambda_s$ (Peard and Gaunt 1995) and $\ln \Lambda_0$ (equation (3.11)), suggests that the $1/d$-expansions have the correct leading behaviour, $\ln \sigma + 1$. At least,

$$\begin{aligned}\ln \lambda_B &= \ln \sigma + 1 - \tfrac{1}{2}\sigma^{-1} - \tfrac{1}{6}\sigma^{-2} - \cdots \\ &> \ln \lambda_0^{(d)} > \ln \Lambda_s^{(d)} > \ln \Lambda_0^{(d)}.\end{aligned} \tag{4.15}$$

Now, if we consider the free energies $F(y; C)$ and $F(z; t)$, we see that

$$F(0; C) = F(0; t) = \ln \Lambda_0 \tag{4.16}$$

$$F(1; C) = \ln \Lambda_s \tag{4.17}$$

$$F(1; t) = \ln \lambda_0. \tag{4.18}$$

So $F^{(d)}(y, z; c, k)$ is likely to be asymptotically correct for $y = 0, 0 \leqslant z \leqslant 1$ and $0 \leqslant y \leqslant 1, z = 0$. Furthermore, compact hypercubic animals will only be important in the collapsed phase, so we may expect $F^{(d)}(y, z; c, k)$ to be asymptotically correct for $y = 0, 0 \leqslant z < z_c^t$ and $0 \leqslant y < y_c^C, z = 0$, where y_c^C and z_c^t are the critical fugacities for the C- and t-models (Derrida and Herrmann 1983, Lam 1988, Gaunt and Flesia 1991). The range over which the expansion $F^{(d)}(y, z; c, k)$ is expected to be valid is for $y < y_c(z)$, where $y_c(z)$ is a curve in the first quadrant joining $(y_c^C, 0)$ and $(0, z_c^t)$. There is no indication whether or not the expansion is valid along this curve. The point (1, 1) must either lie on this curve or outside the region. This raises the interesting possibility that the metric exponent ν for site animals in d-dimensions could be $1/d$.

5. Conclusions

We have derived through fifth order the $1/d$-expansion for the limiting reduced free energy, $F^{(d)}(y, z; c, k)$, of the $\{c, k\}$-model and hence derived two new terms in the $1/d$-expansions for $\ln \lambda_0$ and $\ln \Lambda_0$. In section 2, we show how to use the $1/d$-expansion for the two-variable model to derive the $1/d$-expansions for the c-, k-, s-, q-, C- and t-models. The intermediate results in the derivation of $F^{(d)}(y, z; c, k)$ also allow us to derive new exact enumeration data.

Using exact results we have argued that $F^{(d)}(y, z; c, k)$ is asymptotically correct at least on the lines $y = 0, 0 \leqslant z < z_c^t$ and $0 \leqslant y < y_c^C, z = 0$. We have shown that the $1/d$-expansion for $\ln \lambda_s^{(d)}$ is not asymptotically correct and that the series estimates of Peard and Gaunt (1995) are almost certainly underestimating $\ln \lambda_s$ for all $d \geqslant 4$ and perhaps for $d = 3$. However, we have argued that the expansions for $\ln \lambda_0$, $\ln \Lambda_s$ and $\ln \Lambda_0$ are likely to be correct. In turn, this means that the $1/d$-expansions for the free energies of all the one-variable models will be asymptotically correct only in the expanded phase.

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Appendix A

Partition functions, $Z_n(y, z; c, k)$, see (3.1), for the $\{c, k\}$-model on a d-dimensional simple hypercubic lattice. For $n \leqslant 8$, they are calculated from exact enumeration data alone but, for $9 \leqslant n \leqslant 13$, using the method described in section 3.

$$Z_1(y, z; c, k) = 1$$

$$Z_2(y, z; c, k) = \binom{d}{1}$$

$$Z_3(y, z; c, k) = \binom{d}{1} + 4\binom{d}{2}$$

$$Z_4(y, z; c, k) = \binom{d}{1} + (16 + 4z + y)\binom{d}{2} + 32\binom{d}{3}$$

$$Z_5(y, z; c, k) = \binom{d}{1} + (53 + 32z + 8y)\binom{d}{2} + (324 + 96z + 24y)\binom{d}{3} + 400\binom{d}{4}$$

$$\begin{aligned} Z_6(y, z; c, k) = \binom{d}{1} &+ \left(172 + 160z + 30z^2 + 40y + 14yz + 2y^2\right)\binom{d}{2} \\ &+ \left(2448 + 1512z + 180z^2 + 376y + 84yz + 12y^2\right)\binom{d}{3} \\ &+ (8064 + 2304z + 576y)\binom{d}{4} + 6912\binom{d}{5} \end{aligned}$$

$$Z_7(y,z;c,k) = \binom{d}{1} + \left(568 + 672z + 332z^2 + 168y + 156yz + 22y^2\right)\binom{d}{2}$$
$$+\left(17\,041 + 15\,600z + 4704z^2 + 400z^3 + 3864y + 2208yz + 264yz^2\right.$$
$$\left.+312y^2 + 72y^2z + 8y^3\right)\binom{d}{3}$$
$$+\left(112\,824 + 63\,744z + 9408z^2 + 15\,840y + 4416yz + 624y^2\right)\binom{d}{4}$$
$$+(239\,120 + 62\,720z + 15\,680y)\binom{d}{5} + 153\,664\binom{d}{6}$$

$$Z_8(y,z;c,k) = \binom{d}{1} + \left(1906 + 2712z + 2030z^2 + 336z^3 + 677y + 958yz + 228yz^2\right.$$
$$\left.+134y^2 + 60y^2z + 6y^3\right)\binom{d}{2}$$
$$+\left(116\,004 + 137\,736z + 67\,812z^2 + 15\,096z^3 + 384z^5 + 33\,996y\right.$$
$$+31\,908yz + 10\,080yz^2 + 408yz^4 + 4452y^2 + 2712y^2z$$
$$\left.+212y^2z^3 + 288y^3 + 66y^3z^2 + 12y^4z + y^5\right)\binom{d}{3}$$
$$+\left(1382\,400 + 1141\,248z + 350\,400z^2 + 40\,256z^3 + 282\,216y + 164\,928yz\right.$$
$$\left.+26\,880yz^2 + 23\,040y^2 + 7232y^2z + 768y^3\right)\binom{d}{4}$$
$$+\left(5445\,120 + 2769\,920z + 407\,040z^2 + 688\,640y + 192\,000yz + 26\,880y^2\right)\binom{d}{5}$$
$$+(8257\,536 + 1966\,080z + 491\,520y)\binom{d}{6} + 4194\,304\binom{d}{7}$$

$$Z_9(y,z;c,k) = \binom{d}{1} + \left(6471 + 10\,880z + 9972z^2 + 4064z^3 + 192z^4 + 2708y + 4724yz\right.$$
$$\left.+2776yz^2 + 164yz^3 + 656y^2 + 728y^2z + 62y^2z^2 + 72y^3 + 12y^3z + y^4\right)\binom{d}{2}$$
$$+\left(787\,965 + 1140\,576z + 755\,532z^2 + 287\,280z^3 + 28\,704z^4 + 9216z^5\right.$$
$$+280\,608y + 355\,860yz + 193\,440yz^2 + 24\,720yz^3 + 9792yz^4$$
$$+49\,062y^2 + 51\,552y^2z + 9480y^2z^2 + 5088y^2z^3 + 5328y^3$$
$$\left.+1872y^3z + 1584y^3z^2 + 156y^4 + 288y^4z + 24y^5\right)\binom{d}{3}$$
$$+\left(15\,998\,985 + 17\,116\,800z + 7855\,008z^2 + 1932\,864z^3 + 114\,816z^4\right.$$
$$+24\,576z^5 + 4215\,664y + 3700\,320yz + 1299\,648yz^2 + 98\,880yz^3$$
$$+26\,112yz^4 + 510\,768y^2 + 346\,816y^2z + 37\,920y^2z^2 + 13\,568y^2z^3$$
$$\left.+35\,968y^3 + 7488y^3z + 4224y^3z^2 + 624y^4 + 768y^4z + 64y^5\right)\binom{d}{4}$$
$$+\left(104\,454\,120 + 77\,177\,280z + 22\,232\,640z^2 + 2609\,280z^3 + 19\,098\,480y\right.$$
$$\left.+10\,501\,440yz + 1756\,800yz^2 + 1454\,160y^2 + 469\,120y^2z + 48\,640y^3\right)\binom{d}{5}$$
$$+\left(280\,717\,488 + 128\,770\,560z + 17\,729\,280z^2 + 32\,037\,120y\right.$$
$$\left.+8398\,080yz + 1166\,400y^2\right)\binom{d}{6}$$
$$+(326\,265\,408 + 70\,543\,872z + 17\,635\,968y)\binom{d}{7} + 136\,048\,896\binom{d}{8}$$

$$Z_{10}(y,z;c,k) = \binom{d}{1} + \left(22\,200 + 43\,220z + 46\,004z^2 + 27\,392z^3 + 6062z^4 + 10\,724y\right.$$
$$+21\,844yz + 18\,816yz^2 + 5308yz^3 + 3008y^2 + 4920y^2z$$

$$+2000y^2z^2 + 482y^3 + 378y^3z + 30y^4\big)\tbinom{d}{2}$$
$$+\big(5380\,600 + 9\,167\,304z + 7470\,900z^2 + 3904\,416z^3 + 952\,659z^4$$
$$+167\,760z^5 + 33\,024z^6 + 2248\,620y + 3520\,212yz + 2643\,504yz^2$$
$$+827\,214yz^3 + 177\,060yz^4 + 41\,280yz^5 + 479\,952y^2 + 698\,808y^2z$$
$$+315\,738y^2z^2 + 90\,744y^2z^3 + 25\,608y^2z^4 + 70\,798y^3 + 61\,419y^3z$$
$$+27\,660y^3z^2 + 9888y^3z^3 + 5013y^4 + 4896y^4z + 2448y^4z^2 + 396y^5$$
$$+360y^5z + 24y^6\big)\tbinom{d}{3}$$
$$+\big(180\,558\,848 + 235\,351\,008z + 140\,954\,400z^2 + 52\,264\,576z^3 + 8518\,224z^4$$
$$+1256\,064z^5 + 132\,096z^6 + 57\,763\,584y + 66\,404\,448yz + 35\,316\,960yz^2$$
$$+7383\,456yz^3 + 1325\,088yz^4 + 165\,120yz^5 + 9062\,784y^2 + 9351\,648y^2z$$
$$+2819\,616y^2z^2 + 678\,464y^2z^3 + 102\,432y^2z^4 + 951\,584y^3 + 549\,648y^3z$$
$$+206\,496y^3z^2 + 39\,552y^3z^3 + 44\,976y^4 + 36\,480y^4z + 9792y^4z^2$$
$$+2944y^5 + 1440y^5z + 96y^6\big)\tbinom{d}{4}$$
$$+\big(1839\,569\,920 + 1758\,624\,960z + 736\,709\,440z^2 + 171\,765\,760z^3$$
$$+13\,580\,560z^4 + 1228\,800z^5 + 433\,509\,696y + 348\,072\,000yz$$
$$+116\,214\,720yz^2 + 11\,787\,040yz^3 + 1305\,600yz^4 + 47\,677\,760y^2$$
$$+30\,820\,800y^2z + 4505\,760y^2z^2 + 678\,400y^2z^3 + 3140\,480y^3$$
$$+878\,800y^3z + 211\,200y^3z^2 + 71\,920y^4 + 38\,400y^4z + 3200y^5\big)\tbinom{d}{5}$$
$$+\big(7801\,139\,200 + 5187\,225\,600z + 1371\,264\,000z^2 + 151\,859\,200z^3$$
$$+1285\,228\,800y + 650\,035\,200yz + 102\,950\,400yz^2 + 89\,376\,000y^2$$
$$+27\,340\,800y^2z + 2787\,200y^3\big)\tbinom{d}{6}$$
$$+\big(15\,572\,480\,000 + 6478\,080\,000z + 819\,840\,000z^2 + 1612\,800\,000y$$
$$+389\,760\,000yz + 53\,760\,000y^2\big)\tbinom{d}{7}$$
$$+12\,800\,000(1136 + 224z + 56y)\tbinom{d}{8} + 5120\,000\,000\tbinom{d}{9}$$

$$Z_{11}(y, z; c, k) = \tbinom{d}{1} + \big(76\,884 + 169\,784z + 207\,444z^2 + 148\,728z^3 + 63\,852z^4 + 5696z^5$$
$$+42\,012y + 98\,596yz + 102\,660yz^2 + 56\,496yz^3 + 6032yz^4 + 13\,456y^2$$
$$+26\,756y^2z + 21\,268y^2z^2 + 2912y^2z^3 + 2596y^3 + 3980y^3z + 792y^3z^2$$
$$+310y^4 + 120y^4z + 8y^5\big)\tbinom{d}{2}$$
$$+\big(37\,034\,319 + 72\,525\,600z + 69\,548\,916z^2 + 44\,680\,224z^3 + 17\,341\,872z^4$$
$$+3987\,504z^5 + 919\,680z^6 + 104\,880z^7 + 17\,740\,860y + 32\,773\,380yz$$
$$+30\,367\,248yz^2 + 15\,154\,164yz^3 + 4208\,376yz^4 + 1151\,064yz^5 + 148\,008yz^6$$
$$+4422\,948y^2 + 7967\,040y^2z + 5759\,640y^2z^2 + 2103\,904y^2z^3 + 713\,256y^2z^4$$

$$+106\,104y^2z^5+793\,608y^3+1106\,646y^3z+611\,088y^3z^2+274\,728y^3z^3$$
$$+49\,008y^3z^4+88\,785y^4+101\,160y^4z+67\,800y^4z^2+15\,384y^4z^3$$
$$+7544y^5+9936y^5z+3216y^5z^2+660y^6+408y^6z+24y^7)\binom{d}{3}$$
$$+\big(2017\,563\,224+3087\,285\,504z+2252\,598\,408z^2+1084\,658\,496z^3$$
$$+299\,992\,512z^4+54\,960\,192z^5+9445\,632z^6+839\,040z^7+755\,437\,872y$$
$$+1060\,971\,144yz+735\,484\,416yz^2+261\,507\,552yz^3+57\,870\,624yz^4$$
$$+11\,783\,232yz^5+1184\,064yz^6+143\,308\,436y^2+193\,278\,208y^2z$$
$$+99\,474\,048y^2z^2+28\,986\,944y^2z^3+7278\,272y^2z^4+848\,832y^2z^5$$
$$+19\,340\,928y^3+19\,172\,112y^3z+8465\,856y^3z^2+2793\,728y^3z^3$$
$$+392\,064y^3z^4+1543\,992y^4+1412\,832y^4z+686\,880y^4z^2+123\,072y^4z^3$$
$$+106\,432y^5+100\,288y^5z+25\,728y^5z^2+6640y^6+3264y^6z+192y^7)\binom{d}{4}$$
$$+\big(30\,937\,530\,481+35\,996\,081\,120z+19\,450\,068\,320z^2+6580\,542\,880z^3$$
$$+1145\,908\,480z^4+134\,990\,208z^5+13\,701\,120z^6+8844\,012\,400y$$
$$+9187\,994\,080yz+4468\,344\,240yz^2+999\,695\,680yz^3+142\,520\,640yz^4$$
$$+17\,157\,120yz^5+1245\,882\,880y^2+1176\,870\,320y^2z+380\,726\,080y^2z^2$$
$$+72\,000\,640y^2z^3+10\,645\,120y^2z^4+118\,055\,440y^3+73\,470\,560y^3z$$
$$+21\,340\,800y^3z^2+4107\,520y^3z^3+5923\,120y^4+3635\,520y^4z$$
$$+1015\,680y^4z^2+281\,024y^5+149\,120y^5z+9920y^6\big)\binom{d}{5}$$
$$+\big(194\,498\,568\,156+167\,245\,290\,240z+63\,682\,584\,960z^2$$
$$+13\,587\,383\,040z^3+1122\,984\,960z^4+59\,473\,920z^5+41\,293\,532\,640y$$
$$+30\,190\,824\,960yz+9246\,453\,120yz^2+981\,461\,760yz^3$$
$$+63\,191\,040yz^4+4111\,083\,120y^2+2440\,521\,600y^2z$$
$$+374\,300\,160y^2z^2+32\,834\,560y^2z^3+245\,214\,720y^3+72\,289\,920y^3z$$
$$+10\,222\,080y^3z^2+5830\,080y^4+1858\,560y^4z+154\,880y^5\big)\binom{d}{6}$$
$$+\big(593\,322\,510\,704+357\,772\,800\,000z+86\,356\,596\,480z^2$$
$$+8787\,116\,800z^3+88\,752\,861\,120y+41\,074\,268\,928yz$$
$$+5992\,694\,400yz^2+5613\,943\,104y^2+1584\,499\,840y^2z+159\,371\,520y^3\big)\binom{d}{7}$$
$$+937\,024\,\big(993\,484+377\,216z+43\,680z^2+93\,968y+20\,832yz+2856y^2\big)\binom{d}{8}$$
$$+453\,519\,616\,(1593+288z+72y)\binom{d}{9}+219\,503\,494\,144\binom{d}{10}$$
$$Z_{12}(y,z;c,k)=\binom{d}{1}+\big(268\,350+662\,424z+912\,378z^2+755\,936z^3+435\,330z^4$$
$$+111\,112z^5+4830z^6+163\,494y+433\,922yz+523\,436yz^2+388\,148yz^3$$
$$+119\,378yz^4+5926yz^5+58\,742y^2+135\,796y^2z+146\,048y^2z^2+57\,772y^2z^3$$

$+3452y^2z^4 + 13\,034y^3 + 27\,134y^3z + 15\,582y^3z^2 + 1206y^3z^3 + 2086y^4$
$+2320y^4z + 264y^4z^2 + 151y^5 + 34y^5z + 2y^6)\binom{d}{2}$
$+(257\,091\,447 + 568\,629\,480z + 624\,866\,154z^2 + 467\,409\,000z^3$
$+238\,715\,907z^4 + 80\,060\,424z^5 + 19\,650\,432z^6 + 4958\,880z^7$
$+94\,500z^9 + 138\,748\,804y + 294\,432\,474yz + 318\,594\,384yz^2$
$+209\,651\,542yz^3 + 84\,985\,980yz^4 + 24\,513\,840yz^5 + 7060\,968yz^6$
$+167\,055yz^8 + 39\,370\,626y^2 + 82\,985\,028y^2z + 79\,353\,576y^2z^2$
$+41\,964\,240y^2z^3 + 15\,000\,876y^2z^4 + 5068\,080y^2z^5 + 153\,828y^2z^7$
$+8140\,374y^3 + 15\,078\,957y^3z + 11\,839\,932y^3z^2 + 5661\,432y^3z^3$
$+2330\,616y^3z^4 + 95\,058y^3z^6 + 1191\,223y^4 + 1876\,800y^4z$
$+1361\,688y^4z^2 + 725\,688y^4z^3 + 42\,576y^4z^5 + 132\,228y^5$
$+194\,028y^5z + 150\,000y^5z^2 + 14\,085y^5z^4 + 12\,540y^6 + 18\,744y^6z$
$+3396y^6z^3 + 1080y^7 + 570y^7z^2 + 60y^8z + 3y^9)\binom{d}{3}$
$+(22\,494\,953\,744 + 39\,420\,410\,688z + 33\,669\,058\,848z^2$
$+19\,535\,663\,616z^3 + 7457\,884\,848z^4 + 1932\,787\,968z^5$
$+394\,406\,080z^6 + 71\,993\,664z^7 + 3443\,136z^8 + 756\,000z^9$
$+9620\,030\,144y + 15\,852\,940\,320yz + 13\,280\,745\,120yz^2$
$+6530\,573\,664yz^3 + 2043\,899\,280yz^4 + 490\,531\,968yz^5$
$+102\,111\,456yz^6 + 5393\,280yz^7 + 1336\,440yz^8 + 2121\,438\,880y^2$
$+3464\,676\,032y^2z + 2474\,005\,440y^2z^2 + 1010\,772\,688y^2z^3$
$+299\,844\,064y^2z^4 + 73\,107\,840y^2z^5 + 4368\,192y^2z^6 + 1230\,624y^2z^7$
$+341\,459\,072y^3 + 471\,790\,608y^3z + 286\,854\,864y^3z^2$
$+113\,162\,240y^3z^3 + 33\,573\,312y^3z^4 + 2336\,832y^3z^5 + 760\,464y^3z^6$
$+37\,439\,376y^4 + 45\,880224y^4z + 27\,228\,608y^4z^2 + 10\,448\,544y^4z^3$
$+882\,752y^4z^4 + 340\,608y^4z^5 + 3270\,188y^5 + 3881\,760y^5z$
$+2160\,480y^5z^2 + 236\,480y^5z^3 + 112\,680y^5z^4 + 251\,008y^6$
$+270\,336y^6z + 43\,200y^6z^2 + 27\,168y^6z^3 + 15\,616y^7 + 4864y^7z$
$+4560y^7z^2 + 256y^8 + 480y^8z + 24y^9)\binom{d}{4}$
$+(507\,201\,540\,240 + 691\,805\,061\,120z + 452\,798\,848\,800z^2$
$+195\,805\,808\,000z^3 + 52\,407\,897\,360z^4 + 9325\,311\,360z^5$
$+1352\,772\,160z^6 + 135\,333\,120z^7 + 169\,488\,221\,120y$
$+213\,828\,708\,000yz + 133\,315\,536\,000yz^2 + 45\,924\,089\,760yz^3$
$+9868\,436\,160yz^4 + 1686\,840\,000yz^5 + 192\,347\,520yz^6$

$$
\begin{aligned}
&+28\,730\,073\,120y^2 + 34\,874\,726\,400y^2z + 17\,424\,757\,120y^2z^2\\
&+4901\,520\,320y^2z^3 + 1035\,789\,120y^2z^4 + 138\,038\,400y^2z^5\\
&+3448\,236\,640y^3 + 3329\,263\,600y^3z + 1402\,176\,960y^3z^2\\
&+393\,448\,000y^3z^3 + 63\,542\,400y^3z^4 + 264\,692\,880y^4 + 226\,879\,360y^4z\\
&+95\,424\,320y^4z^2 + 19\,819\,200y^4z^3 + 16\,423\,520y^5 + 13\,720\,960y^5z\\
&+4105\,920y^5z^2 + 894\,560y^6 + 514\,560y^6z + 29\,760y^7)\tbinom{d}{5}\\
&+\big(4548\,861\,718\,272 + 4758\,841\,658\,880z + 2326\,927\,299\,840z^2\\
&+710\,854\,571\,520z^3 + 119\,621\,445\,120z^4 + 12\,373\,857\,792z^5\\
&+1021\,870\,080z^6 + 1171\,471\,401\,600y + 1102\,961\,076\,480yz\\
&+485\,170\,410\,240yz^2 + 105\,004\,723\,200yz^3 + 13\,119\,223\,680yz^4\\
&+1281\,761\,280yz^5 + 148\,856\,217\,600y^2 + 127\,266\,501\,120y^2z\\
&+39\,909\,669\,120y^2z^2 + 6575\,713\,920y^2z^3 + 795\,386\,880y^2z^4\\
&+12\,615\,331\,200y^3 + 7636\,197\,120y^3z + 1912\,832\,640y^3z^2\\
&+306\,708\,480y^3z^3 + 607\,776\,000y^4 + 317\,137\,152y^4z + 75\,755\,520y^4z^2\\
&+23\,707\,200y^5 + 11\,105\,280y^5z + 737\,280y^6\big)\tbinom{d}{6}\\
&+96\,\big(207\,332\,033\,328 + 161\,729\,019\,648z + 56\,076\,356\,448z^2\\
&+10\,860\,151\,680z^3 + 868\,397\,040z^4 + 30\,965\,760z^5 + 39\,996\,819\,072y\\
&+26\,673\,310\,944yz + 7430\,028\,480yz^2 + 763\,751\,520yz^3\\
&+32\,901\,120yz^4 + 3614\,467\,248y^2 + 1953\,567\,840y^2z\\
&+290\,739\,120y^2z^2 + 17\,095\,680y^2z^3 + 194\,018\,160y^3 + 55\,688\,640y^3z\\
&+5322\,240y^3z^2 + 4435\,200y^4 + 967\,680y^4z + 80\,640y^5\big)\tbinom{d}{7}\\
&+55\,296\,\big(844\,047\,744 + 464\,997\,120z + 102\,592\,896z^2 + 9515\,520z^3\\
&+115\,475\,136y + 48\,945\,792yz + 6523\,776yz^2 + 6655\,488y^2\\
&+1718\,528y^2z + 170\,912y^3\big)\tbinom{d}{8}\\
&+31\,850\,496\,\big(1880\,496 + 655\,776z + 69\,552z^2 + 163\,440y\\
&+33\,264yz + 4536y^2\big)\tbinom{d}{9}\\
&+18\,345\,885\,696\,(2160 + 360z + 90y)\tbinom{d}{10} + 10\,567\,230\,160\,896\tbinom{d}{11}
\end{aligned}
$$

$$
\begin{aligned}
Z_{13}(y,z;c,k) = \tbinom{d}{1} &+ \big(942\,649 + 2573\,976z + 3923\,948z^2 + 3718\,712z^3 + 2497\,462z^4\\
&+1047\,168z^5 + 173\,400z^6 + 633\,748y + 1867\,280yz + 2580\,304yz^2\\
&+2239\,824yz^3 + 1136\,672yz^4 + 216\,784yz^5 + 250\,986y^2\\
&+666\,112y^2z + 841\,656y^2z^2 + 551\,136y^2z^3 + 126\,872y^2z^4\\
&+63\,256y^3 + 155\,240y^3z + 147\,808y^3z^2 + 44\,080y^3z^3 + 11\,789y^4
\end{aligned}
$$

$$
\begin{aligned}
&+21\,744y^4z + 9508y^4z^2 + 1392y^5 + 1196y^5z + 68y^6\big)\tbinom{d}{2}\\
&+\big(1799\,484\,655 + 4434\,763\,176z + 5482\,307\,640z^2 + 4643\,410\,280z^3\\
&+2842\,773\,360z^4 + 1258\,287\,744z^5 + 392\,716\,832z^6 + 117\,012\,768z^7\\
&+13\,714\,224z^8 + 3024\,000z^9 + 1079\,653\,764y + 2582\,850\,240yz\\
&+3172\,228\,536yz^2 + 2507\,096\,340yz^3 + 1343\,536\,680yz^4\\
&+490\,376\,640yz^5 + 167\,057\,664yz^6 + 21\,973\,632yz^7 + 5345\,760yz^8\\
&+342\,535\,452y^2 + 820\,712\,904y^2z + 945\,027\,618y^2z^2\\
&+659\,279\,424y^2z^3 + 296\,153\,868y^2z^4 + 119\,519\,664y^2z^5\\
&+17\,859\,576y^2z^6 + 4922\,496y^2z^7 + 79\,384\,424y^3 + 177\,706\,464y^3z\\
&+182\,930\,112y^3z^2 + 108\,958\,712y^3z^3 + 54\,544\,752y^3z^4 + 9453\,384y^3z^5\\
&+3041\,856y^3z^6 + 13\,834\,551y^4 + 28\,266\,216y^4z + 25\,315\,836y^4z^2\\
&+16\,806\,864y^4z^3 + 3497\,736y^4z^4 + 1362\,432y^4z^5 + 1924\,872y^5\\
&+3466\,104y^5z + 3431\,400y^5z^2 + 910\,344y^5z^3 + 450\,720y^5z^4\\
&+215\,000y^6 + 422\,856y^6z + 160\,368y^6z^2 + 108\,672y^6z^3 + 23\,976y^7\\
&+17\,280y^7z + 18\,240y^7z^2 + 864y^8 + 1920y^8z + 96y^9\big)\tbinom{d}{3}\\
&+\big(251\,269\,345\,386 + 495\,378\,785\,088z + 482\,653\,141\,608z^2\\
&+323\,775\,214\,912z^3 + 153\,556\,667\,856z^4 + 52\,802\,975\,232z^5\\
&+13\,518\,644\,864z^6 + 3205\,024\,320z^7 + 405\,765\,888z^8 + 56\,771\,712z^9\\
&+8870\,400z^{10} + 120\,600\,168\,888y + 227\,175\,618\,120yz\\
&+220\,555\,670\,640yz^2 + 134\,958\,792\,888yz^3 + 56\,129\,682\,864yz^4\\
&+16\,816\,064\,160yz^5 + 4554\,979\,296yz^6 + 643\,552\,896yz^7\\
&+99\,707\,232yz^8 + 16\,947\,072yz^9 + 30\,148\,014\,744y^2\\
&+57\,145\,712\,624y^2z + 50\,911\,282\,716y^2z^2 + 27\,568\,653\,536y^2z^3\\
&+10\,163\,267\,040y^2z^4 + 3248\,493\,024y^2z^5 + 522\,232\,128y^2z^6\\
&+91\,198\,464y^2z^7 + 17\,084\,256y^2z^8 + 5553\,870\,320y^3\\
&+9610\,123\,488y^3z + 7688\,473\,536y^3z^2 + 3754\,451\,168y^3z^3\\
&+1479\,318\,624y^3z^4 + 277\,699\,968y^3z^5 + 55\,974\,848y^3z^6\\
&+11\,737\,344y^3z^7 + 751\,959\,474y^4 + 1197\,443\,184y^4z\\
&+877\,614\,688y^4z^2 + 455\,162\,208y^4z^3 + 103\,674\,656y^4z^4\\
&+24\,904\,704y^4z^5 + 5962\,752y^4z^6 + 82\,372\,000y^5 + 121\,001\,216y^5z\\
&+92\,854\,848y^5z^2 + 27\,321\,920y^5z^3 + 8186\,016y^5z^4 + 2299\,200y^5z^5\\
&+7559\,152y^6 + 11\,442\,240y^6z + 4889\,568y^6z^2 + 1961\,088y^6z^3\\
&+671\,904y^6z^4 + 649\,344y^7 + 537\,088y^7z + 326\,976y^7z^2
\end{aligned}
$$

$$
\begin{aligned}
&+145\,344y^7z^3 + 27\,472y^8 + 34\,176y^8z + 22\,080y^8z^2\\
&+1696y^9 + 2112y^9z + 96y^{10}\big)\tbinom{d}{4}\\
&+\big(8200\,792\,614\,025 + 12\,786\,513\,244\,480z + 9760\,249\,011\,880z^2\\
&+5054\,428\,366\,880z^3 + 1783\,731\,159\,040z^4 + 444\,893\,250\,560z^5\\
&+84\,212\,663\,040z^6 + 14\,018\,805\,760z^7 + 1004\,755\,200z^8\\
&+90\,720\,000z^9 + 3124\,790\,530\,240y + 4607\,507\,339\,000yz\\
&+3448\,655\,643\,360yz^2 + 1568\,865\,998\,720yz^3 + 472\,998\,951\,120yz^4\\
&+104\,880\,979\,200yz^5 + 19\,951\,557\,760yz^6 + 1594\,364\,160yz^7\\
&+160\,372\,800yz^8 + 613\,978\,427\,140y^2 + 896\,319\,996\,960y^2z\\
&+592\,977\,780\,640y^2z^2 + 232\,964\,239\,840y^2z^3 + 63\,642\,103\,360y^2z^4\\
&+14\,261\,239\,040y^2z^5 + 1294\,321\,920y^2z^6 + 147\,674\,880y^2z^7\\
&+87\,447\,880\,640y^3 + 112\,217\,094\,080y^3z + 65\,302\,870\,720y^3z^2\\
&+23\,665\,691\,200y^3z^3 + 6513\,140\,160y^3z^4 + 688\,485\,120y^3z^5\\
&+91\,255\,680y^3z^6 + 8803\,927\,760y^4 + 10\,244\,954\,800y^4z\\
&+5579\,820\,960y^4z^2 + 2010\,440\,320y^4z^3 + 257\,100\,480y^4z^4\\
&+40\,872\,960y^4z^5 + 711\,627\,120y^5 + 776\,934\,080y^5z + 411\,484\,480y^5z^2\\
&+67\,768\,320y^5z^3 + 13\,521\,600y^5z^4 + 49\,027\,280y^6 + 50\,866\,560y^6z\\
&+12\,129\,600y^6z^2 + 3260\,160y^6z^3 + 2895\,360y^7 + 1332\,480y^7z\\
&+547\,200y^7z^2 + 68\,160y^8 + 57\,600y^8z + 2880y^9\big)\tbinom{d}{5}\\
&+\big(102\,267\,333\,991\,981 + 125\,353\,072\,932\,000z + 74\,064\,732\,848\,160z^2\\
&+28\,773\,453\,765\,440z^3 + 7156\,757\,201\,280z^4 + 1190\,427\,488\,256z^5\\
&+150\,020\,938\,240z^6 + 13\,779\,924\,480z^7 + 30\,778\,071\,155\,640y\\
&+35\,097\,178\,683\,360yz + 19\,684\,221\,231\,360yz^2\\
&+6306\,320\,081\,280yz^3 + 1267\,246\,137\,600yz^4 + 187\,443\,851\,520yz^5\\
&+19\,678\,752\,000yz^6 + 4698\,580\,928\,880y^2 + 5132\,463\,106\,560y^2z\\
&+2388\,608\,884\,800y^2z^2 + 626\,922\,597\,120y^2z^3 + 114\,601\,656\,960y^2z^4\\
&+14\,133\,876\,480y^2z^5 + 502\,359\,635\,840y^3 + 452\,947\,863\,360y^3z\\
&+177\,087\,319\,680y^3z^2 + 43\,119\,980\,800y^3z^3 + 6493\,305\,600y^3z^4\\
&+35\,599\,946\,400y^4 + 28\,093\,860\,864y^4z + 10\,321\,946\,880y^4z^2\\
&+2017\,532\,160y^4z^3 + 1981\,563\,456y^5 + 1461\,991\,680y^5z\\
&+415\,699\,200y^5z^2 + 93\,853\,120y^6 + 51\,713\,280y^6z + 2960\,640y^7\big)\tbinom{d}{6}\\
&+4\big(154\,249\,705\,997\,947 + 146\,425\,471\,150\,728z + 65\,125\,131\,509\,760z^2\\
&+17\,981\,701\,346\,784z^3 + 2826\,998\,764\,800z^4 + 261\,677\,208\,576z^5
\end{aligned}
$$

$$+17\,371\,361\,280z^6 + 36\,116\,033\,112\,978y + 30\,973\,640\,531\,232yz$$
$$+12\,333\,755\,122\,752yz^2 + 2496\,051\,290\,880yz^3 + 278\,858\,506\,080yz^4$$
$$+21\,823\,226\,880yz^5 + 4164\,253\,950\,276y^2 + 3224\,978\,588\,160y^2z$$
$$+947\,220\,267\,840y^2z^2 + 139\,120\,446\,080y^2z^3 + 13\,544\,119\,680y^2z^4$$
$$+316\,477\,127\,904y^3 + 179\,906\,576\,640y^3z + 39\,937\,151\,520y^3z^2$$
$$+5219\,585\,280y^3z^3 + 14\,157\,877\,440y^4 + 6490\,876\,224y^4z$$
$$+1287\,861\,120y^4z^2 + 473\,178\,496y^5 + 188\,522\,880y^5z + 12\,492\,480y^6\big)\binom{d}{7}$$
$$+2704\,\big(750\,217\,351\,862 + 535\,117\,196\,544z + 169\,631\,681\,408z^2$$
$$+29\,828\,116\,864z^3 + 2238\,261\,760z^4 + 58\,146\,816z^5 + 132\,530\,221\,824y$$
$$+80\,934\,279\,552yz + 20\,506\,617\,600yz^2 + 1979\,931\,520yz^3 + 61\,780\,992yz^4$$
$$+10\,921\,433\,488y^2 + 5374\,510\,848y^2z + 752\,579\,520y^2z^2 + 32\,101\,888y^2z^3$$
$$+528\,487\,232y^3 + 143\,113\,600y^3z + 9993\,984y^3z^2$$
$$+11\,272\,240y^4 + 1817\,088y^4z + 151\,424y^5\big)\binom{d}{8}$$
$$+1827\,904\,\big(2094\,189\,600 + 1061\,144\,352z + 214\,737\,120z^2$$
$$+18\,145\,344z^3 + 263\,755\,944y + 102\,729\,312yz$$
$$+12\,499\,200yz^2 + 13\,905\,720y^2 + 3282\,048y^2z + 323\,232y^3\big)\binom{d}{9}$$
$$+1235\,663\,104\,\big(3348\,185 + 1\,078\,560z + 105\,120z^2 + 268\,920y$$
$$+50\,400yz + 6840y^2\big)\binom{d}{10}$$
$$+835\,308\,258\,304\,(2849 + 440z + 110y)\binom{d}{11} + 564\,668\,382\,613\,504\binom{d}{12}.$$

Appendix B

All non-zero polynomials, $g_{i,c,k}(n)$, see (3.4), for $n \leqslant 6$.

$$g_{1,0,0}(n) = 1$$
$$g_{2,0,0}(n) = (n-2)\big(12 - 7n + 2n^2\big)$$
$$g_{2,0,1}(n) = 4(n-2)(n-3)$$
$$g_{2,1,0}(n) = (n-2)(n-3)$$
$$g_{3,0,0}(n) = \tfrac{1}{6}(n-3)\big(-720 + 1044n - 916n^2 + 459n^3 - 116n^4 + 12n^5\big)$$
$$g_{3,0,1}(n) = 4(n-3)(n-4)\big(45 + 6n - 9n^2 + 2n^3\big)$$
$$g_{3,1,0}(n) = (n-3)(n-4)\big(50 + 5n - 9n^2 + 2n^3\big)$$
$$g_{3,0,2}(n) = 2(n-3)(n-4)(n-5)(21 + 4n)$$
$$g_{3,1,1}(n) = 2(n-3)(n-4)(n-5)(9 + 2n)$$
$$g_{3,2,0}(n) = \tfrac{1}{2}(n-3)(n-4)(n-5)(6 + n)$$

$$g_{4,0,0}(n) = \tfrac{1}{6}(n-4)\big(221\,760 - 120\,552n + 34\,500n^2 - 15\,662n^3 + 9201n^4 - 3913n^5 + 1010n^6 - 140n^7 + 8n^8\big)$$

$$g_{4,0,1}(n) = \tfrac{2}{3}(n-4)(n-5)\big(22\,176 - 4596n - 2112n^2 + 167n^3 + 411n^4 - 128n^5 + 12n^6\big)$$

$$g_{4,1,0}(n) = \tfrac{1}{6}(n-4)(n-5)\big(19\,152 - 3408n - 2550n^2 + 293n^3 + 399n^4 - 128n^5 + 12n^6\big)$$

$$g_{4,0,2}(n) = 2(n-4)(n-5)\big(-21\,840 + 3862n + 1259n^2 - 237n^3 - 42n^4 + 8n^5\big)$$

$$g_{4,1,1}(n) = 2(n-4)(n-5)\big(-10\,416 + 1754n + 573n^2 - 89n^3 - 24n^4 + 4n^5\big)$$

$$g_{4,2,0}(n) = \tfrac{1}{2}(n-4)(n-5)\big(-6720 + 1240n + 358n^2 - 77n^3 - 9n^4 + 2n^5\big)$$

$$g_{4,0,3}(n) = \tfrac{8}{3}(n-4)(n-5)(n-6)\big(-2268 + 30n + 39n^2 + 4n^3\big)$$

$$g_{4,1,2}(n) = 2(n-4)(n-5)(n-6)\big(-1848 - 9n + 33n^2 + 4n^3\big)$$

$$g_{4,2,1}(n) = \tfrac{2}{3}(n-4)(n-5)(n-6)\big(-1568 + 14n + 27n^2 + 3n^3\big)$$

$$g_{4,3,0}(n) = \tfrac{1}{6}(n-4)(n-5)(n-6)\big(-728 + 27n + 12n^2 + n^3\big)$$

$$g_{5,0,0}(n) = \tfrac{1}{360}(n-5)\big(-2863\,123\,200 + 1481\,660\,640n - 483\,329\,520n^2 + 158\,857\,920n^3 - 41\,919\,528n^4 + 11\,632\,070n^5 - 4707\,195n^6 + 1778\,615n^7 - 461\,232n^8 + 73\,640n^9 - 6480n^{10} + 240n^{11}\big)$$

$$g_{5,0,1}(n) = \tfrac{2}{3}(n-5)(n-6)\big(3386\,880 - 1247\,232n - 174\,192n^2 + 113\,859n^3 + 603n^4 - 2549n^5 - 2159n^6 + 978n^7 - 148n^8 + 8n^9\big)$$

$$g_{5,1,0}(n) = \tfrac{1}{6}(n-5)(n-6)\big(4402\,944 - 1535\,520n - 113\,032n^2 + 92\,370n^3 + 6755n^4 - 4004n^5 - 1947n^6 + 966n^7 - 148n^8 + 8n^9\big)$$

$$g_{5,0,2}(n) = \tfrac{1}{3}(n-5)(n-6)\big(-29\,060\,640 + 10\,009\,092n + 771\,888n^2 - 585\,587n^3 - 6480n^4 + 19\,027n^5 + 312n^6 - 596n^7 + 48n^8\big)$$

$$g_{5,1,1}(n) = \tfrac{1}{3}(n-5)(n-6)\big(-13\,214\,880 + 4439\,508n + 421\,968n^2 - 275\,323n^3 - 3930n^4 + 8123n^5 + 450n^6 - 316n^7 + 24n^8\big)$$

$$g_{5,2,0}(n) = \tfrac{1}{12}(n-5)(n-6)\big(-7680\,960 + 2626\,584n + 271\,164n^2 - 180\,758n^3 - 351n^4 + 5866n^5 - 105n^6 - 140n^7 + 12n^8\big)$$

$$g_{5,0,3}(n) = \tfrac{8}{3}(n-5)(n-6)(n-7)\big(-926\,640 + 161\,982n + 31\,095n^2 - 5625n^3 - 421n^4 + 34n^5 + 8n^6\big)$$

$$g_{5,1,2}(n) = 2(n-5)(n-6)(n-7)\big(-825\,120 + 135\,060n + 27\,458n^2 - 4259n^3 - 437n^4 + 22n^5 + 8n^6\big)$$

$$g_{5,2,1}(n) = \tfrac{2}{3}(n-5)(n-6)(n-7)\big(-704\,880 + 118\,838n + 22\,511n^2 - 3842n^3 - 314n^4 + 21n^5 + 6n^6\big)$$

$$g_{5,3,0}(n) = \tfrac{1}{6}(n-5)(n-6)(n-7)\big(-318\,240 + 58\,148n + 9520n^2 - 2005n^3 - 94n^4 + 13n^5 + 2n^6\big)$$

$$g_{5,0,4}(n) = \tfrac{4}{3}(n-5)(n-6)(n-7)(n-8)\big(-299\,835 + 9312n + 4875n^2 + 408n^3 + 16n^4\big)$$

$$g_{5,1,3}(n) = \tfrac{4}{3}(n-5)(n-6)(n-7)(n-8)\big(-118\,935 + 2118n + 2001n^2 + 186n^3 + 8n^4\big)$$

$$g_{5,2,2}(n) = \tfrac{1}{3}(n-5)(n-6)(n-7)(n-8)\big(-184\,050 + 4187n + 3064n^2 + 279n^3 + 12n^4\big)$$

$$g_{5,3,1}(n) = \tfrac{1}{3}(n-5)(n-6)(n-7)(n-8)\big(-37\,980 + 1403n + 628n^2 + 51n^3 + 2n^4\big)$$

$$g_{5,4,0}(n) = \tfrac{1}{24}(n-5)(n-6)(n-7)(n-8)\big(-27\,000 + 1542n + 435n^2 + 30n^3 + n^4\big)$$

$$g_{5,0,5}(n) = 1024(n-5)(n-6)(n-7)n^2$$

$$g_{5,1,4}(n) = 1088(n-5)(n-6)(n-7)n^2$$

$$g_{5,2,3}(n) = \tfrac{1696}{3}(n-5)(n-6)(n-7)n^2$$

$$g_{5,3,2}(n) = 176(n-5)(n-6)(n-7)n^2$$

$$g_{5,4,1}(n) = 32(n-5)(n-6)(n-7)n^2$$

$$g_{5,5,0}(n) = \tfrac{8}{3}(n-5)(n-6)(n-7)n^2$$

$$g_{6,0,0}(n) = \tfrac{1}{360}(n-6)\big(1482\,126\,750\,720 - 798\,510\,586\,752n + 215\,216\,294\,688n^2 - 47\,483\,000\,112n^3 + 10\,387\,896\,720n^4 - 2430\,232\,048n^5 + 560\,296\,250n^6 - 127\,877\,327n^7 + 37\,825\,992n^8 - 12\,444\,153n^9 + 3197\,694n^{10} - 553\,688n^{11} + 60\,016n^{12} - 3664n^{13} + 96n^{14}\big)$$

$$g_{6,0,1}(n) = \tfrac{1}{90}(n-6)(n-7)\big(15\,601\,766\,400 - 10\,254\,781\,440n - 260\,867\,520n^2 + 1185\,643\,680n^3 - 94\,765\,896n^4 - 63\,271\,318n^5 + 10\,169\,645n^6 + 281\,570n^7 + 588\,191n^8 - 359\,272n^9 + 72\,920n^{10} - 6720n^{11} + 240n^{12}\big)$$

$$g_{6,1,0}(n) = \tfrac{1}{360}(n-6)(n-7)\big(-9488\,793\,600 - 1228\,515\,840n - 2306\,738\,880n^2 + 1629\,839\,040n^3 - 170\,304\,336n^4 - 47\,671\,378n^5 + 6174\,005n^6 + 1152\,950n^7 + 450\,071n^8 - 346\,552n^9 + 72\,440n^{10} - 6720n^{11} + 240n^{12}\big)$$

$$g_{6,0,2}(n) = \tfrac{1}{3}(n-6)(n-7)\big(-6939\,820\,800 + 3256\,452\,000n - 78\,057\,672n^2 - 184\,885\,732n^3 + 21\,066\,810n^4 + 4749\,219n^5 - 702\,576n^6 - 104\,087n^7 + 11\,954n^8 + 3700n^9 - 680n^{10} + 32n^{11}\big)$$

$$g_{6,1,1}(n) = \tfrac{1}{3}(n-6)(n-7)\big(-3584\,528\,640 + 1621\,619\,424n - 45\,381\,816n^2 - 83\,434\,460n^3 + 8677\,270n^4 + 2360\,385n^5 - 321\,938n^6 - 46\,753n^7 + 3512n^8 + 2144n^9 - 352n^{10} + 16n^{11}\big)$$

$$g_{6,2,0}(n) = \tfrac{1}{12}(n-6)(n-7)\big(-2207\,969\,280 + 989\,353\,728n - 34\,311\,520n^2 - 48\,266\,008n^3 + 5032\,556n^4 + 1512\,606n^5 - 221\,549n^6 - 31\,280n^7 + 4929n^8 + 742n^9 - 164n^{10} + 8n^{11}\big)$$

$$g_{6,0,3}(n) = \tfrac{4}{9}(n-6)(n-7)\big(16\,258\,682\,880 - 7062\,251\,328n + 393\,295\,464n^2 + 236\,025\,720n^3 - 30\,186\,642n^4 - 3473\,208n^5 + 649\,625n^6 + 16\,097n^7 - 3188n^8 - 524n^9 + 48n^{10}\big)$$

$$g_{6,1,2}(n) = \tfrac{1}{3}(n-6)(n-7)\big(14\,449\,881\,600 - 6158\,986\,560n + 302\,142\,240n^2 + 210\,765\,960n^3 - 25\,131\,642n^4 - 3073\,233n^5 + 494\,588n^6 + 22\,547n^7 - 2216n^8 - 596n^9 + 48n^{10}\big)$$

$$g_{6,2,1}(n) = \tfrac{1}{9}(n-6)(n-7)\big(11\,835\,901\,440 - 5078\,119\,104n + 249\,098\,976n^2 + 179\,095\,964n^3 - 22\,291\,950n^4 - 2491\,966n^5 + 449\,463n^6 + 13\,232n^7 - 2007n^8 - 420n^9 + 36n^{10}\big)$$

$$g_{6,3,0}(n) = \tfrac{1}{36}(n-6)(n-7)\big(5020\,392\,960 - 2200\,262\,976n + 115\,391\,088n^2 + 80\,673\,272n^3 - 11\,077\,174n^4 - 1028\,219n^5 + 235\,503n^6 + 827n^7 - 1157n^8 - 104n^9 + 12n^{10}\big)$$

$$g_{6,0,4}(n) = \tfrac{2}{3}(n-6)(n-7)(n-8)\big(1794\,081\,960 - 510\,150\,186n - 7414\,491n^2 + 12\,190\,617n^3 - 398\,307n^4 - 90\,279n^5 - 786n^6 + 352n^7 + 32n^8\big)$$

$$g_{6,1,3}(n) = \tfrac{4}{3}(n-6)(n-7)(n-8)\big(767\,055\,960 - 209\,928\,966n - 4328\,913n^2 + 4849\,221n^3 - 97\,093n^4 - 38\,997n^5 - 752n^6 + 140n^7 + 16n^8\big)$$

$$g_{6,2,2}(n) = \tfrac{1}{3}(n-6)(n-7)(n-8)\big(1211\,332\,320 - 332\,725\,392n - 5820\,234n^2 + 7566\,579n^3 - 180\,823n^4 - 58\,405n^5 - 1015n^6 + 210n^7 + 24n^8\big)$$

$$g_{6,3,1}(n) = \tfrac{1}{3}(n-6)(n-7)(n-8)\big(248\,125\,680 - 70\,117\,668n - 669\,620n^2 + 1585\,935n^3 - 55\,961n^4 - 11\,419n^5 - 67n^6 + 44n^7 + 4n^8\big)$$

$$g_{6,4,0}(n) = \tfrac{1}{24}(n-6)(n-7)(n-8)\big(172\,307\,520 - 50\,852\,352n + 70\,140n^2 + 1145\,124n^3 - 59\,027n^4 - 7227n^5 + 77n^6 + 31n^7 + 2n^8\big)$$

$$g_{6,0,5}(n) = \tfrac{8}{15}(n-6)(n-7)(n-8)\big(-1188\,380\,160 + 308\,727\,756n - 6779\,724n^2 - 1967\,865n^3 - 65\,430n^4 + 9815n^5 + 456n^6 + 16n^7\big)$$

$$g_{6,1,4}(n) = \tfrac{2}{3}(n-6)(n-7)(n-8)\big(-927\,210\,240 + 230\,775\,804n - 2634\,288n^2 - 1607\,223n^3 - 69\,576n^4 + 7835n^5 + 408n^6 + 16n^7\big)$$

$$g_{6,2,3}(n) = \tfrac{4}{3}(n-6)(n-7)(n-8)\big(-225\,232\,920 + 56\,414\,172n - 706\,498n^2 - 386\,905n^3 - 17\,205n^4 + 1941n^5 + 99n^6 + 4n^7\big)$$

$$g_{6,3,2}(n) = \tfrac{1}{3}(n-6)(n-7)(n-8)\big(-257\,273\,280 + 66\,637\,488n - 1261\,966n^2 - 439\,831n^3 - 17\,195n^4 + 2355n^5 + 105n^6 + 4n^7\big)$$

$$g_{6,4,1}(n) = \tfrac{1}{30}(n-6)(n-7)(n-8)\big(-417\,154\,320 + 113\,465\,532n - 3215\,240n^2 - 700\,877n^3 - 21\,790n^4 + 4140n^5 + 150n^6 + 5n^7\big)$$

$$g_{6,5,0}(n) = \tfrac{1}{120}(n-6)(n-7)(n-8)\big(-120\,059\,280 + 34\,595\,148n - 1376\,372n^2 - 190\,615n^3 - 4320n^4 + 1290n^5 + 36n^6 + n^7\big)$$

$$g_{6,0,6}(n) = 512(n-6)(n-7)(n-8)(n-9)n^2(135+8n)$$

$$g_{6,1,5}(n) = 128(n-6)(n-7)(n-8)(n-9)n^2(655+42n)$$

$$g_{6,2,4}(n) = \tfrac{16}{3}(n-6)(n-7)(n-8)(n-9)n^2(9725+628n)$$

$$g_{6,3,3}(n) = \tfrac{32}{3}(n-6)(n-7)(n-8)(n-9)n^2(1900+119n)$$

$$g_{6,4,2}(n) = 16(n-6)(n-7)(n-8)(n-9)n^2(320+19n)$$

$$g_{6,5,1}(n) = \tfrac{16}{3}(n-6)(n-7)(n-8)(n-9)n^2(145+8n)$$

$$g_{6,6,0}(n) = \tfrac{8}{3}(n-6)(n-7)(n-8)(n-9)n^2(20+n).$$

Appendix C

$1/d$-expansions for the free energy of one-variable site animal models.

c-model. $y = \mathrm{e}^{\beta}$

$$F^{(d)}(y;c) = \ln\sigma + 1 + \big(-\tfrac{1}{2} + \tfrac{1}{2}y\big)\sigma^{-1} + \big(-\tfrac{8}{3} + \tfrac{3}{8}y^2\big)\sigma^{-2} + \big(-\tfrac{85}{12} + \tfrac{3}{2}y + \tfrac{55}{24}y^2 + \tfrac{13}{12}y^3\big)\sigma^{-3} + \big(-\tfrac{931}{20} - \tfrac{21}{2}y + \tfrac{7}{3}y^2 + \tfrac{53}{6}y^3 + \tfrac{841}{192}y^4 + \tfrac{1}{6}y^5\big)\sigma^{-4} + \big(-\tfrac{2777}{10} - \tfrac{69}{2}y + \tfrac{391}{24}y^2 + \tfrac{289}{6}y^3 + \tfrac{7243}{160}y^4 + \tfrac{8221}{480}y^5 + \tfrac{3}{4}y^6\big)\sigma^{-5} + \mathrm{O}\big(\sigma^{-6}\big).$$

s-model. $\theta = \mathrm{e}^{-2\beta}$

$$F^{(d)}(\theta;s) = 2(d-1)\beta + \ln\sigma + 1 + \big(-\tfrac{5}{2} + \tfrac{5}{2}\theta\big)\sigma^{-1} + \big(-\tfrac{13}{6} - \tfrac{13}{2}\theta + \tfrac{51}{8}\theta^2\big)\sigma^{-2} + \big(-\tfrac{191}{12} + \tfrac{159}{4}\theta - \tfrac{805}{8}\theta^2 + \tfrac{895}{12}\theta^3\big)\sigma^{-3} + \big(-\tfrac{139}{5} - 307\theta + \tfrac{1983}{2}\theta^2 - \tfrac{5175}{4}\theta^3 + \tfrac{79\,721}{192}\theta^4 + \tfrac{361}{2}\theta^5\big)\sigma^{-4} + \big(-\tfrac{1585}{3} + \tfrac{17\,075}{16}\theta - \tfrac{159\,599}{24}\theta^2 + \tfrac{433\,781}{24}\theta^3 - \tfrac{647\,235}{32}\theta^4 + \tfrac{931\,543}{160}\theta^5 + \tfrac{9025}{4}\theta^6\big)\sigma^{-5} + \mathrm{O}\big(\sigma^{-6}\big).$$

q-model. $\zeta = \mathrm{e}^{-\beta}$

$$F^{(d)}(\zeta;q) = 2(d-1)\beta + \ln\sigma + 1 + \big(-\tfrac{5}{2} + 2\zeta + \tfrac{1}{2}\zeta^2\big)\sigma^{-1} + \big(-\tfrac{13}{6} - 5\zeta + 3\zeta^2 + \tfrac{3}{2}\zeta^3 + \tfrac{3}{8}\zeta^4\big)\sigma^{-2} + \big(-\tfrac{191}{12} + 33\zeta - \tfrac{243}{4}\zeta^2 + \tfrac{46}{3}\zeta^3 + \tfrac{141}{8}\zeta^4 + \tfrac{89}{12}\zeta^5 + \tfrac{13}{12}\zeta^6\big)\sigma^{-3} + \big(-\tfrac{139}{5} - \tfrac{499}{2}\zeta + \tfrac{1243}{2}\zeta^2 - 483\zeta^3 - \tfrac{437}{4}\zeta^4 + \tfrac{147}{4}\zeta^5 + \tfrac{407}{4}\zeta^6 + \tfrac{158}{3}\zeta^7 + \tfrac{2569}{192}\zeta^8 + 2\zeta^9 + \tfrac{1}{6}\zeta^{10}\big)\sigma^{-4}$$

$$+\left(-\tfrac{1585}{3}+\tfrac{3597}{4}\zeta-\tfrac{69\,665}{16}\zeta^2+\tfrac{52\,043}{6}\zeta^3-\tfrac{14\,857}{3}\zeta^4-\tfrac{213\,209}{120}\zeta^5\right.$$
$$\left.-\tfrac{7309}{48}\zeta^6+\tfrac{22\,495}{24}\zeta^7+\tfrac{23\,129}{32}\zeta^8+\tfrac{21\,809}{80}\zeta^9+\tfrac{30\,301}{480}\zeta^{10}+\tfrac{19}{2}\zeta^{11}+\tfrac{3}{4}\zeta^{12}\right)\sigma^{-5}$$
$$+\mathrm{O}(\sigma^{-6}).$$

t-model. $z=\mathrm{e}^{\beta}$

$$F^{(d)}(z;t)=\ln\sigma+1+\left(-\tfrac{5}{2}+2z\right)\sigma^{-1}+\left(-\tfrac{13}{6}-5z+\tfrac{9}{2}z^2\right)\sigma^{-2}$$
$$+\left(-\tfrac{191}{12}+33z-\tfrac{135}{2}z^2+\tfrac{130}{3}z^3\right)\sigma^{-3}$$
$$+\left(-\tfrac{139}{5}-\tfrac{499}{2}z+679z^2-\tfrac{1501}{2}z^3+\tfrac{953}{4}z^4+64z^5\right)\sigma^{-4}$$
$$+\left(-\tfrac{1585}{3}+\tfrac{3597}{4}z-4522z^2+\tfrac{62\,831}{6}z^3-\tfrac{20\,333}{2}z^4+\tfrac{56\,321}{20}z^5+752z^6\right)\sigma^{-5}$$
$$+\mathrm{O}(\sigma^{-6}).$$

C-model. $y=\mathrm{e}^{\beta}$

$$F^{(d)}(y;C)=\ln\sigma+1+\left(-\tfrac{5}{2}+\tfrac{1}{2}y\right)\sigma^{-1}+\left(-\tfrac{13}{6}-\tfrac{3}{2}y+\tfrac{3}{8}y^2\right)\sigma^{-2}$$
$$+\left(-\tfrac{191}{12}+\tfrac{27}{4}y-\tfrac{41}{8}y^2+\tfrac{13}{12}y^3\right)\sigma^{-3}$$
$$+\left(-\tfrac{139}{5}-\tfrac{115}{2}y+45y^2-\tfrac{39}{2}y^3+\tfrac{457}{192}y^4+\tfrac{1}{6}y^5\right)\sigma^{-4}$$
$$+\left(-\tfrac{1585}{3}+\tfrac{2687}{16}y-\tfrac{7919}{24}y^2+\tfrac{6097}{24}y^3-\tfrac{2747}{32}y^4+\tfrac{3661}{480}y^5+\tfrac{3}{4}y^6\right)\sigma^{-5}$$
$$+\mathrm{O}(\sigma^{-6}).$$

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